SOME INVARIANTS PRESERVED BY ISOMORPHIMS OF TABLES OF MARKS

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Tables of marks.

Definition 1. Let G, Q be finite groups. Let $\mathfrak{C}(G)$ be the family of all conjugacy classes of subgroups of G. We usually assume that the elements of $\mathfrak{C}(G)$ are ordered non-decreasingly. Let ψ be a function from $\mathfrak{C}(G)$ to $\mathfrak{C}(Q)$. Given a subgroup H of G, we denote by H' any representative of $\psi([H])$. We say that ψ is an *isomorphism between the tables of marks of* G *and* Q if ψ is a bijection and if $\#((Q/K')^{H'}) = \#((G/K)^H)$ for all subgroups H, K of G.

The matrix whose H, K-entry is $\#((G/K)^H)$ is called the **table of marks** of G (where H, K run through all the elements in $\mathfrak{C}(G)$). Some authors define the table of marks of G as the transpose of the previous matrix (for instance, that is how GAP defines it).

The **Burnside ring** of G, denoted $B(G)$, is the subring of $\mathbb{Z}^{\mathfrak{C}(G)}$ spanned by the columns of the table of marks of G. It is easy to see that if G and Q have isomorphic tables of marks, then they have isomorphic Burnside rings; the converse is an open problem.

Preserved attributes.

Theorem 2. *Let G, Q be finite groups with isomorphic tables of marks. Let K, H denote subgroups of G, and let K', H' denote representatives in their respective conjugacy classes of subgroups under the isomorphism between their tables of marks. Then we have that:*

1. $G' = Q, (1_G)' = 1_Q, |G| = |G'|, |H| = |H'|$, $\alpha(H,K) = \alpha(H',K'), \beta(H,K) = \beta(H',K')$, $|N_G(H)| = |N_Q(H')|$.

2. *The subgroup H is normal in G if and only if H' is normal in Q. In this case, G/H and Q/H' have isomorphic tables of marks.*

3. *If $K \leq H$ and at least one of these two subgroups is normal in G, then $K' \leq H'$ for any choice of K' and H'.*

4. *If K and H are normal subgroups of G, then $(K \cap H)' = K' \cap H'$ and $(KH)' = K'H'$. In particular, two normal subgroups with trivial intersection correspond to two normal subgroups with trivial intersection. Furthermore, if $G = K \times H$, then $Q = K' \times H'$, K and K' have isomorphic tables of marks, and H and H' have isomorphic tables of marks.*

5. *The subgroup H is maximal in G if and only if H' is maximal in Q.*

6. *If G is a p-group, then $socle(Z(G))' = socle(Z(Q))$.*

7. *The Frattini subgroups correspond, that is, $\Phi(G)' = \Phi(Q)$.*

8. *The group G is nilpotent if and only if Q is nilpotent. However, there are non-isomorphic p-groups with isomorphic tables of marks.*

9. *For any divisor d of the order of H, the number of subgroups of H of order d is preserved; in particular, the total number of subgroups of H is preserved.*

10. *The subgroup H is cyclic if and only if H' is cyclic.*

11. *If H is isomorphic to the quaternion group of order 8, then H' is isomorphic to H.*

12. *If G is abelian then $G \cong Q$.*

13. *The commutator subgroups correspond, that is,* $[G, G]' = [Q, Q]$. *Moreover, the abelianized groups are isomorphic, that is,* $G/[G, G] \cong Q/[Q, Q]$.

14. *If* G *is isomorphic to* S_n *for some* $n \geq 5$, *then* Q *is isomorphic to* G.

15. *The subgroup* H *is elementary abelian if and only if* H' *is elementary abelian.*

Counterexamples.

We wrote software in GAP to go through the library of SmallGroups searching for the first instance of non-isomorphic groups with isomorphic tables of marks. The first known example before this computation was Thévenaz's pair of groups of order $5 * 11^2 = 625$. The smallest example has order 96. Let G be `SmallGroup(96,108)`, and let Q be `SmallGroup(96,114)`

Consider the following file written in GAP:

```
G := SmallGroup(96,108);
Q := SmallGroup(96,114);
#------------------------------
```

```
testlattice := function(g,h)
  lat := LatticeSubgroups(g);
  conj := ConjugacyClassesSubgroups(lat);
  lath := LatticeSubgroups(h);
  conjh := ConjugacyClassesSubgroups(lath);
  for n in [1..Size(conj)] do
    subg := ClassElementLattice(conj[n],1);
    subh := ClassElementLattice(conjh[n],1);
     if not(IsAbelian(subg)=IsAbelian(subh)) then
       Print("Subgroup number ",n,", Order ",Order(s
              "  ",IsAbelian(subg),"  ",IsAbelian(su
     fi;
  od;
end;
#-----------------------------------------
subn := function(g,n)
# Return a representative of the n-th conjugcy clas
# of g.
  return ClassElementLattice(ConjugacyClassesSubgro
           (LatticeSubgroups(g))[n],1);
end;
```

After loading this file, we checked that the tables of marks of G and Q are identical:

```
gap> Read("testiso");
gap> MatTom(TableOfMarks(G))=MatTom(TableOfMarks(Q)
true
```

This means that there is an isomorphism between the tables of marks of G and Q. Moreover, with the default tables of marks assigned by GAP, this isomorphism maps the n-th conjugacy class of subgroups of G to the n-th conjugacy class of subgroups of Q. We wonder whether the centres of G and Q correspond under this isomorphism.

```
gap> Size(Centre(G));
8
gap> Size(Centre(Q));
4
```

This proves that the centres of G and Q cannot correspond under this or any other isomorphism between their tables of marks (since such isomorphisms must preserve the order of the subgroups). Next we wonder whether abelian subgroups of G must necessarily correspond with abelian subgroups of Q.

The function `testlattice(G,Q)` runs through all the conjugacy classes of subgroups of G and Q (which are the same length), and it tests whether the corresponding subgroups are both abelian or both non-abelian. When it finds a pair of corresponding subgroups that do not match, it prints them on the screen, displaying their order and whether they are abelian or not.

```
gap> testlattice(G,Q);
Subgroup number 36, Order 16  true  false
Subgroup number 37, Order 16  false  true
Subgroup number 38, Order 16  false  true
Subgroup number 40, Order 16  true  false
Subgroup number 58, Order 48  true  false
gap>
```

These are the only corresponding subgroups which are neither both abelian nor both non-abelian. Notice that there is exactly one abelian subgroup of G of order 48 which does not correspond to an abelian subgroup of Q (in fact, according to GAP, Q has no abelian subgroups of order 48). This means that G has exactly one more abelian subgroup of order 48 than Q, so there is no isomorphism between the tables of marks of G and Q that preserves abelian subgroups.

Finally, we show that the table of marks cannot provide enough information to determine the normalizer of a subgroup. Consider the function **subn(g,n)**, which returns a representative of the n-th conjugacy class of subgroups of the group g.

```
gap> Normalizer(G,subn(G,2))=subn(G,58);
true
gap> Normalizer(Q,subn(Q,2))=subn(Q,58);
false
```

In both cases we had a subgroup in the second conjugacy class of subgroups; in G, its normalizer was the (only normal) subgroup in the 58-th conjugacy class, but in Q, the corresponding subgroup is not the normalizer.

We can summarize all this in the following result.

Theorem 3. *Let G and Q be finite groups with isomorphic tables of marks, and let $H \mapsto H'$ denote an isomorphism between their tables of marks. We have that*

1. *H and H' may not be isomorphic.*

2. *Even if H is abelian, H' need not be abelian.*

3. *H and H' may have different tables of marks.*

4. *Even if $K \times L = H$, it may not be possible to find K', L' and H' such that $K' \times L' = H'$.*

5. *Even if K is normal in H, it may not be possible to choose K' and H' such that K' is normal in H'*

6. *Given H, the table of marks does not determine which subgroup of G is the normalizer of H in G.*

7. *$Z(G)'$ may not equal $Z(Q)$.*

Proof. Let G be `SmallGroup(96,108)` and Q be `SmallGroup(96,114)`.

1. This was known since Thévenaz's example, but it is also a consequence of our counterexample.

2. This is immediate.

3. This follows from the previous item and the fact that the table of marks determines an abelian group up to isomorphism.

4. If this were true, since cyclic subgroups correspond, it would follow that abelian subgroups map to abelian subgroups.

5. The subgroup `subn(G,2)` is a counterexample.

6. The subgroup `subn(G,2)` is again a counterexample.

7. Their orders are different.

□