Minimal groups with isomorphic Tables of Marks

(Work in progress)

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Introduction.

We had previously used GAP to construct the smallest example of two non-isomorphic groups with isomorphic tables of marks. We have already constructed those two groups, which have isomorphic tables of marks but are not isomorphic (*"Two non-isomorphic groups of order 96 with isomorphic tables of marks and non-corresponding centres and abelian subgroups"*, to appear in *Communications in Algebra*). We are now trying to prove that they are indeed the minimal such counterexample.

Tables of marks.

Definition 1. Let G, Q be finite groups. Let CG be the family of all conjugacy classes of subgroups of G. We usually assume that the elements of CG are ordered non-decreasingly. Let ψ be a function from CG to CQ. Given a subgroup H of G, we denote by H' any representative of $\psi([H])$. We say that ψ is an *isomorphism between the tables of marks of* G *and* Q if ψ is a bijection and if $\#(K'^{H'}) = \#(K^H)$ for all subgroups H, K of G.

An isomorphism between tables of marks preserves the order of the subgroups, the order or their normalizers, it sends cyclic groups to cyclic groups and elementary abelian groups to elementary abelian groups. It also sends the derived subgroup of G to the derived subgroup of Q, maximal subgroups of G to maximal subgroups of Q, and the Frattini subgroup of G to the Frattini subgroup of Q. Since it preserves normal groups, it is easy to check that if two groups G and Q have isomorphic tables of marks and G is abelian or simple, then G and Q must be isomorphic groups. If G is a direct product, so is Q, and their corresponding factors have isomorphic tables of marks.

However, an isomorphis of tables of marks may not preserve abelian subgroups, and it may not send the centre of G to the centre of Q.

Counterexamples.

Let S_3 be the symmetric group or order 6, let C_8 be the cyclic group or order 8, generated by x, and let C_2 be the cyclic group of order 2, generated by y. Let δ be the composition of the non-trivial homomorphism from S_3 to C_2 followed by the non-trivial homomorphism from C_2 to C_8. Let W denote the group $S_3 \times C_8$; let α be the automorphism of W given by $\alpha(\lambda, x^i) = (\lambda, x^i\delta(\lambda))$, and let β be the automorphism of W given by $\beta(\lambda, x^i) = (\lambda, x^{5i}\delta(\lambda))$. Since α has order two, we can define the group G as the semidirect product of W with C_2 by α, that is, in G we have that $y(\lambda, x^i)y = \alpha(\lambda, x^i)$. Similarly, we define the group Q as the semidirect product of W and C_2 by β.

The groups G and Q have order 96, and have isomorphic tables of marks, but the centre of G has order 8 and the centre of Q has order 4. Moreover, G has an abelian subgroup of order 48, whereas Q has no abelian subgroup of that order.

Proving their minimality.

Let $A(n)$ denote the number of non-abelian groups of order n up to isomorphism. Using GAP we can list the values of n and $A(n)$ for n from 2 to 95:

2: 0; 3: 0; 4: 0; 5: 0; 6: 1; 7: 0; 8: 2; 9: 0; 10: 1; 11: 0; 12: 3; 13: 0; 14: 1; 15: 0; 16: 9; 17: 0; 18: 3; 19: 0; 20: 3; 21: 1; 22: 1; 23: 0; 24: 12; 25: 0; 26: 1; 27: 2; 28: 2; 29: 0; 30: 3; 31: 0; 32: 44; 33: 0; 34: 1; 35: 0; 36: 10; 37: 0; 38: 1; 39: 1; 40: 11; 41: 0; 42: 5; 43: 0; 44: 2; 45: 0; 46: 1; 47: 0; 48: 47; 49: 0; 50: 3; 51: 0; 52: 3; 53: 0; 54: 12; 55: 1; 56: 10; 57: 1; 58: 1; 59: 0; 60: 11; 61: 0; 62: 1; 63: 2; 64: 256; 65: 0; 66: 3; 67: 0; 68: 3; 69: 0; 70: 3; 71: 0; 72: 44; 73: 0; 74: 1; 75: 1; 76: 2; 77: 0; 78: 5; 79: 0; 80: 47; 81: 10; 82: 1; 83: 0; 84: 13; 85: 0; 86: 1; 87: 0; 88: 9; 89: 0; 90: 8; 91: 0; 92: 2; 93: 1; 94: 1; 95: 0;

$A(n) = 0$ for the following 40 values of n: 2, 3, 4, 5, 7, 9, 11, 13, 15, 17, 19, 23, 25, 29, 31, 33, 35, 37, 41, 43, 45, 47, 49, 51, 53, 59, 61, 65, 67, 69, 71, 73, 77, 79, 83, 85, 87, 89, 91, 95.

$A(n) = 1$ for the following 20 values of n: 6, 10, 14, 21, 22, 26, 34, 38, 39, 46, 55, 57, 58, 62, 74, 75, 82, 86, 93, 94.

$A(n) = 2$ for the following 7 values of n: 8, 27, 28, 44, 63, 76, 92

$A(n) = 3$ for the following 9 values of n: 12, 18, 20, 30, 50, 52, 66, 68, 70.

$A(n) = 5$ for $n = 42$ and $n = 78$.

$A(n) = 8$ for $n = 90$.

$A(n) = 9$ for $n = 16$ and $n = 88$.

$A(n) = 10$ for $n = 36$, $n = 56$ and $n = 81$.

$A(n) = 11$ for $n = 40$ and $n = 60$.

$A(n) = 12$ for $n = 24$ and $n = 54$.

$A(n) = 13$ for $n = 84$.

$A(n) = 44$ for $n = 32$ and $n = 72$.

$A(n) = 47$ for $n = 48$ and $n = 80$.

$A(n) = 256$ for $n = 64$.

Some cases are easy:

Theorem 2. *Let* n *be a prime number, or the square of a prime number, or a number of the form* pq *where* $p > q$ *are primes and* q *does not divide* $p - 1$. *Then all groups of order* n *are abelian.*

This accounts for all the values n such that $A(n) = 0$ except for $n = 45$, which is easy to prove directly.

Theorem 3. *Let n be a number of the form pq where $p > q$ are primes and q divides $p - 1$. Then there is exactly one isomorphism class of non-abelian groups of order n.*

This accounts for all the values n such that $A(n) = 1$ except for $n = 75$, which is easy to prove directly, since the only non-abelian group of order 75 must be the only non-trivial semidirect product $(C_5 \times C_5) \rtimes C_3$.

The case $A(n) = 2$

The seven possible values of n are: 8, 27, 28, 44, 63, 76, 92.

Here we must not only count all possible isomorphism classes of non-abelian groups, but we must also prove they have non-isomorphic tables of marks.

The case $n = 8$ is well-known (we can only have the quaternions and the dihedral group). Their tables of marks are not isomorphic, because all the subgroups of the quaternions are normal, which is not true of the dihedral group.

The case $n = 27$ is in the literature (for example, in Suzuki's introduction to the theory of groups). One of the groups has a cyclic subgroup of order 9, but the other group has no cyclic subgroups of that order.

The cases when n equals 28, 44, 76 and 92 are all of the form $4p$ with p a prime number larger than 4 and congruent to 3 modulo 4 (namely, 7, 11, 19 and 23). Here we have that the group must be a semidirect product, either $C_p \rtimes C_4$ or $C_p \rtimes (C_2 \times C_2)$. The Sylow 2-subgroups cannot correspond under an isomorphism of tables of marks.

The case $n = 63$ must be a semidirect product, either $C_7 \rtimes C_9$ or $C_7 \rtimes (C_3 \times C_3)$. The Sylow 3-subgroups cannot correspond under an isomorphism of tables of marks.

The case $A(n) = 3$

The nine possible values of n are: 12, 18, 20, 30, 50, 52, 66, 68, 70.

The case $n = 12$ is in the literature. The only non-abelian groups of order 12 are A_4, $S_3 \times C_2$ (which is $\cong D_{12}$) and the only non-trivial semidirect product $C_3 \rtimes C_4$. The third group has a Sylow 2-subgroup isomorphic to C_4, and the second group has a normal subgroup of order 6, so no two of these three groups have isomorphic tables of marks.

The cases when $n = 18$ and $n = 50$ are of the form $2p^2$ with $p = 3$ and $p = 5$. The group has to be either the only non-trivial semidirect product $C_{p^2} \rtimes C_2$ or one of the two non-trivial semidirect products $(C_p \times C_p) \rtimes C_2$. The first group has a cyclic Sylow p-subgroup. In the case $(C_p \times C_p) \rtimes C_2$, since $C_p \times C_p$ is a vector space over the field with p elements, its automorphisms of order 2 are easily computed: one of them has an invariant one-dimensional subspace, and the other one does not. One possible semidirect product has C_p as a direct summand, and the other one does not, so their tables of marks cannot be isomorphic.

The cases when n equals 20, 52 and 68 are all of the form $4p$ with p a prime number larger than 4 and congruent to 1 modulo 4 (namely, 5, 13 and 17). Here the possible groups are the only non-trivial semidirect product $C_p \rtimes (C_2 \times C_2)$, and the only two non-trivial semidirect products $C_p \rtimes C_4$. The Sylow 2-subgroup of the first group cannot correspond to C_4 under an isomorphism of tables of marks. In one of the semidirect products $C_p \rtimes C_4$, C_4 acts on C_p as the involution $x \mapsto x^{-1}$, so C_2 centralizes C_p, so the group has a normal subgroup of order 2; but in the other semidirect product, C_4 acts by an automorphism of C_P of order 4, so C_2 cannot centralize C_p, so this groups has no normal subgroup of order 2.

The cases when n equals 30, 66 and 70 are all of the form $2pq$ with p, q primes, $p > q > 2$ and q does not divide $p - 1$. First we observe that all groups of order less than 100 are soluble (except for A_5, which has order 60). By P. Hall's Theorem, a soluble group G of order $2pq$ has precisely one normal subgroup of order pq, so G is a semidirect product $H \rtimes C_2$ where H is a group of order pq. Since q does not divide $p-1$, H is abelian, so G is $(C_p \times C_q) \rtimes C_2$. The automorphism group of $C_p \times C_q$ has three elements of order 2, so there are at most three non-abelian choices for the group G. Precisely one of these groups has a direct factor isomorphic to C_q, and precisely another of these groups has a direct factor isomorphic to C_p, so neither two of the three groups can have isomorphic tables of marks.

The case $A(n) = 5$

There are two possible values for n, namely, 42 and 78. Both are numbers of the form $2pq$ with $p > q$ primes and q divides $p-1$ (actually, $q = 3$ in both cases). Since these groups are soluble, by Hall's Theorem there is precisely one normal subgroup H of order pq. Here we have two possibilities: H could be the only non-abelian group of order pq, or H could be C_{pq} (groups from these two cases cannot have isomorphic tables of marks). If H is cyclic, there are three elements of order two in its automorphism group, so there are three possible non-trivial semidirect products $H \rtimes C_2$: one has C_p as a direct factor (but not C_q), the other has C_q as a direct factor (but not C_p), and the other has no such direct factors, so they cannot have isomorphic tables of marks.

Now assume that H is non-abelian. Note that H has a normal subgroup C_p and p subgroups C_q. An automorphism σ of H of order two must fix C_p setwise, and permute the p different C_q, so it fixes one of the C_q setwise, and acts here and on C_p either as the involution $x \mapsto x^{-1}$ or the identity. Moreover, the fixed points under σ (which form a normal subgroup of H) are trivial unless σ equals the identity map. Since H can be generated by a generator of C_p and a generator of one of the C_q's, there is only one automorphism of H of order 2, so one possible group is $H \times C_2$, and the other is the only non-trivial semidirect product $H \rtimes C_2$, and these two groups cannot have isomorphic tables of marks.

Remaining 16 cases

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