Two non-isomorphic groups of order 96 with isomorphic tables of marks and non-corresponding centres and abelian subgroups

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Abstract

We construct two non-isomorphic groups G and Q of order 96 which have isomorphic tables of marks, but such that the centre of G has order 8 and the centre of Q has order 4. We also note that G has an abelian subgroup of order 48, whereas Q has no abelian subgroup of that order. This also leads us to conclude that this isomorphism of tables of marks does not preserve normalizers.

Short title: Tables of marks. [1]

1 Introduction.

Groups with isomorphic tables of marks may not be isomorphic groups (as proved by Thévenaz in [7]), but one still expects them to have many attributes in common. Indeed, if G and Q are groups with isomorphic tables of marks, then their chief series are determined (see [3]), so that they have isomorphic composition factors, and they also have isomorphic Burnside rings (the converse is still an open problem, put forward in [5]); if two groups have isomorphic Burside rings and one of them is abelian/Hamiltonian/minimal simple, then the two groups are isomorphic (see [6]), and a similar result is known for several families of simple groups (see [4]). The reader can find in [2] a short survey of results known about the properties that an isomorphism of tables of marks preserves: for example, it preserves cyclic/elementary abelian subgroups, and maps the commutator subgroup to the commutator

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subgroup. In that article we also show code written in GAP (see [1]) which we used to construct two non-isomorphic groups of order 96 with isomorphic tables of marks, but with centres of different sizes and different numbers of abelian subgroups of order 48. In this paper we explicitly construct these groups, prove they have isomorphic tables of marks, analyse their centres and their subgroups of order 48.

In Section 2 we give the basic definitions and notation we shall use though-out this paper. In Section 3 we construct two special non-isomorphic groups of order 96 with isomorphic tables of marks, and conclude from this example that one cannot determine either abelian subgroups or the center of the group from the table of marks; we also observe that this problem is impossible to fix, in other words, there is neither an isomorphism between the tables of marks of these groups that preserves abelian subgroups, nor an isomorphism that makes the centres of these groups correspond. The same groups show that one cannot determine the normalizer of a subgroup from the table of marks, and that normalizers are not preserved by isomorphisms of tables of marks.

2 Tables of marks.

Definition 2.1. Let G, Q be finite groups. Let $\mathfrak{C}(G)$ be the family of all conjugacy classes of subgroups of G. We usually assume that the elements of $\mathfrak{C}(G)$ are ordered non-decreasingly. Let ψ be a function from $\mathfrak{C}(G)$ to $\mathfrak{C}(Q)$. Given a subgroup H of G, we denote by H' any representative of $\psi([H])$. We say that ψ is an *isomorphism between the tables of marks of G and Q* if ψ is a bijection and if $\#(K'^{H'}) = \#(K^H)$ for all subgroups H, K of G.

Note that $\#(K^H) = \frac{|N_G(H)|}{|K|}\alpha(H,K)$, where $\alpha(H,K)$ equals the number of subgroups of K which are G-conjugate to H. If ψ is an isomorphism between the tables of marks of G and Q and we denote $\psi(H)$ by H', we have that $|H'| = |H|$, $|N_Q(H')| = |N_G(H)|$, and $alpha(H,K))\alpha(H',K')$ for all H, K in $\mathfrak{C}(G)$; in fact, the previous conditions are an equivalent definition of an isomorphism between the tables of marks of G and Q. The matrix whose H,K-entry is $\#(K^H)$ is called the **table of marks** of G (where H, K run through all the elements in $\mathfrak{C}(G)$). Some authors define the table of marks of G as the transpose of the previous matrix (for instance, that is how GAP defines it). Note that this matrix is defined up to an ordering of the elements of $\mathfrak{C}(G)$, so that the groups G and Q have isomorphic tables of marks if and only if it is possible to rearrange the elements of $\mathfrak{C}(G)$ and/or $\mathfrak{C}(Q)$ so that G and Q have identical tables of marks.

The **Burnside ring** of G, denoted $B(G)$, is the subring of $\mathbb{Z}^{\mathfrak{C}(G)}$ spanned by the columns of the table of marks of G. It is easy to see that if G and Q have isomorphic tables of marks, then they have isomorphic Burnside rings; the converse is an open problem (see [5]).

3 Counterexamples.

Let S_3 be the symmetric group or order 6, and denote its elements $\sigma = \sigma_1 = (123), \sigma_2 = (132), \tau = \tau_1 = (12), \tau_2 = (13), \tau_3 = (2,3)$; we write λ for an arbitrary element of S_3. Let C_8 be the cyclic group or order 8, generated by x, and let C_2 be the cyclic group of order 2, generated by y. Let δ be the composition of the non-trivial homomorphism from S_3 to C_2 followed by the non-trivial homomorphism from C_2 to C_8, that is, $\delta(\tau_i) = x^4$ and $\delta(\sigma_j) = 1$. Let W denote the group $S_3 \times C_8$; let α be the automorphism of W given by $\alpha(\lambda, x^i) = (\lambda, x^i\delta(\lambda))$, and let β be the automorphism of W given by $\beta(\lambda, x^i) = (\lambda, x^{5i}\delta(\lambda))$. Since α has order two, we can define the group G as the semidirect product of W with C_2 by α, that is, in G we have that $y(\lambda, x^i)y = \alpha(\lambda, x^i)$. Similarly, we define the group Q as the semidirect product of W and C_2 by β; in Q we have that $y(\lambda, x^i)y = \beta(\lambda, x^i)$. We shall denote the elements of both G and Q as $\lambda x^i y^j$.

Note that in G, x and y commute, and the centre of G is therefore the subgroup generated by x, which is a subgroup of order 8; however, x and y do not commute in Q, and the centre of Q is the subgroup generated by x^2, which is a subgroup of order 4. In particular, we also have that G and Q are non-isomorphic groups of order 96. Notice also that the centraliser of σ in G is the subgroup generated by σ, x and y, which is an abelian subgroup of order 48. On the other hand, the only elements in Q of order 3 are σ and σ_2, and their centraliser is again the subgroup generated by σ, x and y, which is a non-abelian subgroup of order 48, so Q has no abelian subgroup of order 48.

Let us prove that G and Q have isomorphic tables of marks. Let H be the subgroup (of both G and Q) generated by x^4. It is easy to see that H is contained in the centres of G and Q, and that G/H is isomorphic to Q/H (since x equals x^5 in the quotient groups). This gives a natural isomorphism between the tables of marks of G/H and Q/H. By the correspondence theorem, we can lift this to G and Q to obtain a bijection between the conjugacy classes of subgroups which contain x^4, and this bijection preserves order, order of normaliser and number of conjugates containing a given subgroup. Moreover, this bijection sends a subgroup K of G to itself (viewed now as a subgroup of Q). It suffices to show that this assignment (that is, mapping a subgroup K of G to itself as a subgroup of Q) can be extended to the subgroups of G and Q which do not contain x^4.

In both G and Q there are exactly four conjugacy classes of elements of order two, which are x^4, y, x^4y, $(12), (13), (23), (12)x^4, (13)x^4, (23)x^4$ and $(12)x^2y, (13)x^2y, (23)x^2y, (12)x^6y, (13)x^6y, (23)x^6y$. These last three conjugacy classes together with the 3-cycle σ account for all the conjugacy classes of nontrvial subgroups of G and Q which do not contain x^4. Therefore, the assignment that maps K to itself gives a complete isomorphism between the tables of marks of G and Q, since it is not so difficult to see that it also preserves the orders of the normalizers, the orders of the subgroups and the number of conjugates containing a given subgroup. Finally, note that this isomorphism of tables of marks sends the subgroup generated by y to itself, but the normalizer of this subgroup (which is also its centralizer) in G is the subgroup generated by σ, x and y, but x does not normalize y in Q.

We can summarize all this in the following result.

Theorem 3.1. *Let G and Q be finite groups with isomorphic tables of marks, and let $H \mapsto H'$ denote an isomorphism between their tables of marks. We have that*

1. *H and H' may not be isomorphic.*
2. *Even if H is abelian, H' need not be abelian.*
3. *H and H' may have different tables of marks.*
4. *Even if $K \times L = H$, it may not be possible to find K', L' and H' such that $K' \times L' = H'$.*
5. *Even if K is normal in H, it may not be possible to choose K' and H' such that K' is normal in H'*
6. *Given H, the table of marks does not determine which subgroup of G is the normalizer of H in G.*

Proof. Let G and Q be the groups we defined earlier.

1. This was known since Thévenaz's example, but it is also a consequence of our counterexample.
2. Take the abelian subgroup of G of order 48.
3. This follows from the previous item and the fact that the table of marks determines an abelian group up to isomorphism.
4. If this were true, since cyclic subgroups correspond, it would follow that abelian subgroups map to abelian subgroups.
5. The subgroup of G generated by y is a counterexample, because it is a normal subgroup of its normalizer (which is itself a normal subgroup of G), but in Q it has no conjugate which is a normal subgroup of the corresponding subgroup of order 48.
6. The subgroup generated by y is again a counterexample.

$\square$

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